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Exact solutions of the complex modified Korteweg–de Vries equation

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Abstract. Firstly, some similarity reductions of the complex modified Korteweg–de Vries equation (CMKdV), which arises in the asymptotic interpretation of one-dimensional plane-wave propogation in a quadratic micropolar medium are discussed. Although it is not a soliton equation solvable by inverse scattering transformation, its similarity reductions obtained by the use of Lie group methods are of mathematical interest. Secondly, the Painlevé analysis developed by Weiss *et al* for nonlinear partial differential equations is applied to the CMKdV equation, and the data obtained by the truncation technique yield some analytical solutions of the ordinary modified Korteweg–de Vries equation and travelling-wave solutions of the CMKdV equation which are also solutions of the similarity reduction obtained by classical Lie group analysis.

1. Introduction

Quasilinear parabolic equations, or nonlinear reaction–diffusion systems, arise in the modelling of phenomena in physics, chemistry, biology and other applied sciences. The complex modified Korteweg–de Vries equation (CMKdV)

$$w_t + \alpha(|w|^2 w)_x + \beta w_{xxx} = 0 \tag{1}$$

arises in the asymptotic investigation of electrostatic waves in a magnetized plasma, and in the asymptotic interpretation of one-dimensional plane-wave propagation in a quadratic micropolar medium [1].

In [2] a solitary wave solution

$$A(x,t) = \alpha \operatorname{sech}\left[\left(\frac{a}{2b}^{1/2}\alpha(x) - \tfrac{1}{2}\alpha^2 t\right) + \delta\right] \tag{2}$$

is obtained, and it is shown that CMKdV equation does not pass the Painlevé test given by Weiss *et al* [3] for complete integrability.

In this paper we first study the similarity reductions of the CMKdV equation. The classical method for finding similarity reductions of a given partial differential equation is to use the Lie group method of infinitesimal transformations, originally developed by Lie [4]. Though the method is entirely algorithmic, it often involves a large amount of tedious algebra and auxiliary calculations which are virtually unmanageable manually. Recently, symbolic manipulation programs have been developed, especially in Mathematica, Macsyma and Reduce. In order to facilitate the determination of the associated similarity reductions, we used the package SPDE in the computer algebra system Reduce.

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Secondly we will see that Painlevé analysis is a powerful tool for the construction of explicit solutions and Lie–Bäcklund transformations. For integrable equations it also helps to find Lax pairs and recursion operators. It also plays an important role in the study of the chaotic behaviour of nonlinear partial differential equations.

2. Classical similarity reductions

Since $|w|$ in (1) brings some difficulty in the calculations, we first let $w = u + iv$ and separate the real and imaginary parts in (1) and obtain the system

$$\begin{aligned} &u_t + \alpha(u^3)_x + \beta u_{xxx} + \alpha(uv^2)_x = 0 \\ &v_t + \alpha(v^3)_x + \beta v_{xxx} + \alpha(vu^2)_x = 0. \end{aligned} \tag{3}$$

Then using the package SPDE in the computer algebra system Reduce we find that the system (3) admits a Lie group with generators

$$\begin{aligned} &X_1 = \frac{\partial}{\partial x} \qquad X_2 = \frac{\partial}{\partial t} \qquad X_3 = v\,\frac{\partial}{\partial u} - u\,\frac{\partial}{\partial v} \\ &X_4 = x\,\frac{\partial}{\partial x} + 3t\,\frac{\partial}{\partial t} - u\,\frac{\partial}{\partial u} - v\,\frac{\partial}{\partial v}. \end{aligned} \tag{4}$$

The non-vanishing commutators are

$$[X_1, X_4] = X_1 \qquad [X_2, X_4] = 3X_2. \tag{5}$$

The following list presents the one-parameter groups of point transformations, the form of the invariant solutions and similarity reductions corresponding to the generators X_1, X_2, X_3 and X_4:

Generator	Point transformations	Invariant solution	Similarity reduction	
X_1 :	$x^* = x + \varepsilon$ $t^* = t$ $w^* = w$	$f(t)$	$f' = 0$	(6)
X_2 :	$x^* = x$ $t^* = t + \varepsilon$ $w^* = w$	$f(x)$	$bf'' + a\|f\|^2 f = D$	(7)
X_3 :	$x^* = x$ $t^* = t$ $w^* = w\ \mathrm{e}^{i\varepsilon}$	No invariant solution		(8)
X_4 :	$x^* = \varepsilon x$ $t^* = \varepsilon^3 t$ $w^* = w/\varepsilon$	$f(xt^{-1/3})$	$bf'' - \frac{1}{3}zf + a\|f\|^2 f = D$	(9)
while				
$\frac{1}{c}X_1 + X_2$:	$x^* = x + \frac{1}{c}\varepsilon$ $t^* = t + \varepsilon$ $w^* = w$	$f(x - ct)$	$bf'' - cf + a\|f\|^2 f = D.$	(10)

3. Painlevé test for PDEs

Since the formulation of the Painlevé tests, there has been considerable interest in using the Painlevé property as a means of determining whether given equations, both partial and ordinary differential equations, are integrable. To apply the test to a partial differential equation we use the theory of complex functions with several complex variables.

The major difference between analytic functions of one complex variable and several complex variables is that, in general, the singularities of a function of several complex variables cannot be isolated [6]. If $f = f(z_1, \ldots, z_n)$ is a meromorphic function of n complex variables ($2n$ real variables), the singularities of f occur along analytic manifolds of (real) dimension $2n - 2$. These manifolds are determined by conditions of the form

$$\phi(z_1, \ldots, z_n) = 0 \tag{11}$$

where ϕ is an analytic function of $(z_1, \ldots, z_n)$ in a neighbourhood of the manifold.

With reference to the above, we say that *a partial differential equation has the Painlevé property when the solutions of the PDE are single-valued about the movable singularity manifolds*. For partial differential equations we require that the solution be a *single-valued functional* of the data, i.e. *arbitrary functions*. This is a formal property and not a restriction on the data itself.

To verify if a PDE has the Painlevé property we introduce a method for expanding a solution of a nonlinear PDE about a movable, singular manifold (11). Let $u = u(z_1, \ldots, z_n)$ be a solution of the PDE and assume that

$$u = \phi^p \sum_{j=0}^{\infty} u_j \phi^j \tag{12}$$

where ϕ and

$$u_j = u_j(z_1, \ldots, z_n) \tag{13}$$

are analytic functions of $(z_1, \ldots, z_n)$ in a neighbourhood of the manifold (11). Substitution of (12) into the PDE determines the possible values of p and defines the recursion relations for u_j, $j = 0, 1, 2, \ldots$. When p is a negative integer and (12) is a valid and general expansion about the manifold (11), then the solution has a single-valued representation about (11). If this representation is valid for all allowed movable singularity manifolds, then the PDE has the Painlevé property. For a specific PDE it is necessary to identify all possible values for p and then find what the form of the resulting *phi* series [7] is.

A point that should be emphasized is that the *phi* series for nonlinear PDE contains a lot of information about the PDE. For equations which have the Painlevé property a method has been developed for finding the Lax pairs and Bäcklund transformations [8–10]. An outline and an application of the *singular manifold method* is presented in section 4. For equations that do not have the Painlevé property, it is still possible to obtain single-valued expansions by specializing the arbitrary functions that appear in the *phi* series expansions. This specialization leads to a system of partial differential equations for the formally arbitrary data. For specific systems, and it is also conjectured in general, these equations are integrable. The form of the resulting reduction enables the identification of integrable reductions of the original systems [12].

Now we are going to illustrate the nature of the Painlevé test on the complex modified Korteweg–de Vries equation (1).

4. Painlevé analysis for the CMKdV Equation

Let $\phi(x,t)=0$ be the solution singularity manifold of (3) and

$$u=\phi^{p}\sum_{j=0}^{\infty}u_j\phi^j \qquad v=\phi^{p}\sum_{j=0}^{\infty}v_j\phi^j. \tag{14}$$

Substituting (14) into (3) we have $p=-1$,

$$u_0^2+v_0^2=-\frac{2\beta}{\alpha}\phi_x^2 \tag{15}$$

$$u_1=-\frac{u_{0x}}{\phi_x}+\frac{u_0\phi_{xx}}{2\phi_x^2} \qquad v_1=-\frac{v_{0x}}{\phi_x}+\frac{v_0\phi_{xx}}{2\phi_x^2} \tag{16}$$

and

$$\begin{aligned}6\beta\phi_x^4u_2&=\phi_t\phi_xu_0+\beta\left(3\phi_x^2u_{0xx}-6\phi_x\phi_{xx}u_{0x}-2\phi_x\phi_{xxx}u_0+\tfrac{9}{2}\phi_{xx}^2u_0\right)\\ 6\beta\phi_x^4v_2&=\phi_t\phi_xv_0+\beta\left(3\phi_x^2v_{0xx}-6\phi_x\phi_{xx}v_{0x}-2\phi_x\phi_{xxx}v_0+\tfrac{9}{2}\phi_{xx}^2v_0\right).\end{aligned} \tag{17}$$

The resonance points are $j=-1,\ 0,\ 3$ and $j=4$. Clearly the resonance point $j=-1$ corresponds to the free singularity manifold function $\phi(x,t)$. At the resonance $j=0$, we have relation (15); hence u_0 or v_0 is arbitrary. For $j=3$, the recurrance relation is satisfied identically. Hence u_3 and v_3 are arbitrary. However, for $j=4$ we see that only one of the u_4 or v_4 is arbitrary. To write a general solution to this system one needs six arbitrary functions. However, the number of the arbitrary quantities here is five. Therefore the system of quasilinear partial differential equations (3) does not pass the painlevé test for PDEs [2], but this failure does not prevent us from deriving some valuable results from the data obtained from this analysis.

5. Exact Solutions of the CMKdV Equation

Let us truncate the series in (12) at the second term and assume that $u_j=0,\ \ j\geqslant 2$. Then

$$u=\frac{u_0}{\phi}+u_1 \qquad v=\frac{v_0}{\phi}+v_1. \tag{18}$$

If we let $u_2=v_2=0$ in (17) we get

$$\begin{aligned}&\phi_t\phi_xu_0+\beta(3\phi_x^2u_{0xx}-6\phi_x\phi_{xx}u_{0x}-2\phi_x\phi_{xx}u_0+\tfrac{9}{2}\phi_{xx}^2u_0)=0\\ &\phi_t\phi_xv_0+\beta(3\phi_x^2v_{0xx}-6\phi_x\phi_{xx}v_{0x}-2\phi_x\phi_{xx}v_0+\tfrac{9}{2}\phi_{xx}^2v_0)=0.\end{aligned} \tag{19}$$

Substituting (18) in (3), we obtain a necessary condition to have a solution of the form (18):

$$\begin{aligned}&u_{1t}+\alpha(u_1^3)_x+\beta u_{1xxx}+\alpha(u_1v_1^2)_x=0\\ &v_{1t}+\alpha(v_1^3)_x+\beta v_{1xxx}+\alpha(v_1u_1^2)_x=0\end{aligned} \tag{20}$$

i.e. $u_1,\ v_1$ must be a solution of the original system of PDEs (3). Therefore relation (18) can be taken as a auto-Bäcklund transformation that relates two solutions $u,\ v$ and $u_1,\ v_1$ of (3) if $u_1,\ v_1$ given by (16), and $u_0,\ v_0$ interrelated by (15) satisfy (19) and (20).

Hence, solving (15), (19) and (20) for $u_0(t,x)$, $v_0(t,x)$ and $\phi(t,x)$ one obtains a particular solution of the CMKdV equation through (16) and (18).

Although this is a nonlinear system of partial differential equations and its solution is even more difficult than the original one, we are interested only in some particular solutions.

Let us define two invariant functions under the group $\mathcal{H}$ of homographic transformations

$$\phi \longrightarrow \frac{a\phi + b}{c\phi + d} \qquad ad - bc \neq 0 \tag{21}$$

$$S = \{\phi, x\} = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \tag{22}$$

called the Schwarzian derivative, and

$$C = -\frac{\phi_t}{\phi_x} \tag{23}$$

which has the dimension of a velocity, and a non-invariant ratio:

$$L = -\frac{\phi_{xx}}{2\phi_x}. \tag{24}$$

The two elementary invariants S and C are linked by the compatibility condition $(\phi_t)_{xxx} = (\phi_{xxx})_t$ which reads [12]

$$S_t + C_{xxx} + 2C_x S + CS_x = 0 \tag{25}$$

when expressed in terms of C and S.

Let us define two more functions $U(t,x)$, $V(t,x)$ by

$$U(t,x) = \frac{u_0}{\phi_x} \qquad V(t,x) = \frac{v_0}{\phi_x}. \tag{26}$$

In terms of L, the two invariants S, C and new functions U, V, the expressions in (15), (16) become

$$U^2 + V^2 = -\frac{2\beta}{\alpha} \tag{27}$$

$$u_1 = -U_x + LU \qquad v_1 = -V_x + LV \tag{28}$$

and the expressions in (19) transform into

$$\begin{aligned} &3\beta U_{xx} + (\beta S - C)U = 0 \\ &3\beta V_{xx} + (\beta S - C)V = 0. \end{aligned} \tag{29}$$

Multiplying the first of these equations by U_x and the second by V_x and adding the resulting equations, we get

$$\left[(U_x)^2 + (V_x)^2\right]_x = 0 \tag{30}$$

and integrating once we obtain

$$(U_x)^2 + (V_x)^2 = \frac{2\beta}{\alpha} D(t) \tag{31}$$

where $D(t)$ is an arbitrary function of t. While multiplying the first of these equations by U and the second by V and adding the resulting equations, one gets

$$\beta S - C = -3\beta D(t). \tag{32}$$

Hence the expressions in (29) become

$$\begin{aligned} &U_{xx} - D(t)U = 0 \\ &V_{xx} - D(t)V = 0 \end{aligned} \tag{33}$$

and have solutions which satisfy (27):

$$U(t,x)=\sqrt{\frac{-2\beta}{\alpha}}\cos(\sqrt{-D(t)}x+\Omega(t))$$
$$V(t,x)=\sqrt{\frac{-2\beta}{\alpha}}\sin(\sqrt{-D(t)}x+\Omega(t)) \tag{34}$$

where $\Omega(t)$ is a function of t which neads to be determined.

Now equation (20) in terms of the new functions is

$$L(U_t+SU_x)-(U_t+SU_x)_x-\tfrac{1}{2}C_xU_x=0$$
$$L(V_t+SV_x)-(V_t+SV_x)_x-\tfrac{1}{2}C_xV_x=0. \tag{35}$$

On the other hand, adding the equations in (35) one obtains

$$\frac{\mathrm{d}D(t)}{\mathrm{d}t}+3\beta D(t)S_x=0 \tag{36}$$

and hence

$$D(t)S_{xx}=0. \tag{37}$$

Now there are two possibilities; $D(t)\equiv 0$ or $S_{xx}\equiv 0$. In what follows we are going to discuss these two cases serarately.

Case 1. $D(t)\equiv 0$

In this case U and V are real constants such that

$$U^2+V^2=-\frac{2\beta}{\alpha}. \tag{38}$$

If $U=0$ or $V=0$, equations (18), (26) and (28) imply that $u\equiv 0$ or $v\equiv 0$. Then one of the equations in (3) is satisfied automatically. The other equation as well as the original CMKdV equation (1) reduce to the usual modified Korteweg–de Vries equation, namely

$$\Phi_t+\alpha(\Phi^3)_x+\beta\Phi_{xxx}=0. \tag{39}$$

If $U\cdot V\not\equiv 0$ *then the symmetry of the equations implies that*

$$U^2=V^2=-\frac{\beta}{\alpha}\phi_x^2 \tag{40}$$

and hence one gets

$$\pm U=V=\varepsilon \qquad \varepsilon=\sqrt{-\frac{\beta}{\alpha}} \tag{41}$$

$$\pm u_1=v_1=-\frac{\varepsilon}{2}\frac{\phi_{xx}}{\phi_x}=\varepsilon L. \tag{42}$$

In this paper we shall consider only the plus sign in $\pm$. The negative sign leads to the complex conjugate of the solution obtained in the case of the positive sign.

In this case by (32) we have

$$C-\beta\, S=0 \tag{43}$$

and in terms of the two invariants S, C and the non-invariant ratio L, the system in (20) reduces into a single equation

$$L_t-2\beta(L^3)_x+\beta\, L_{xxx}=0 \tag{44}$$

or equivalently

$$2(C-\beta S)L^2 - 2(C-\beta S)_x L + (C-\beta S)_{xx} + S(C-\beta S) = 0. \tag{45}$$

In view of (43), the left-hand side of (45) vanishes identically, and hence imposes no further restriction on S and C. Therefore the only conditions on S and C are relation (43) and the compatibility condition (25).

Substitution of (43) in (25) leads to the Korteweg–de Vries equation

$$S_t + \beta S_{xxx} + 3\beta S\, S_x = 0. \tag{46}$$

The above analysis reveals that for any solution of the KdV equation (46) in S one obtains the solutions of the equation (22) for $\phi(t,x)$ and hence a solution of the CMKdV equation (3). However to obtain the solutions of (22), one needs the following two lemmas about the differential equations written in terms of the Schwarzian derivatives [13].

Lemma 1. Let Ψ_1 and Ψ_2 be two linearly independent solutions of the equation

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}z^2} + P(z)\Psi = 0 \tag{47}$$

which are defined and holomorphic on some simply connected domain D in the complex plane. Then $W(z) = \Psi_1(z)/\Psi_2(z)$ satisfies the equation

$$\{W; z\} = 2P(z) \tag{48}$$

at all points of D where $\Psi_2(z) \neq 0$. Conversely, if $W(z)$ is a solution of (48), holomorphic in some neighbourhood of $z_0 \in D$, then one can find two linearly independent solutions $\Psi_1(z)$ and $\Psi_2(z)$ of (47) such that $W(z) = \Psi_1(z)/\Psi_2(z)$.

Lemma 2. The Schwartzian derivative is invariant under fractional linear transformation acting on the first argument, namely,

$$\left\{\frac{aW+b}{cW+d}; z\right\} = \{W; z\} \qquad ad - bc \neq 0 \tag{49}$$

where $a,\ b,\ c,\ d$ are constants.

Let us now consider two simple special cases.

A. Solutions for constant S. The constant functions $S = \pm 2\lambda^2$ with λ a constant are solutions of the Korteweg–de Vries equation (46).

For $S = -2\lambda^2$. Equation (22) becomes

$$S = \{\phi; x\} = -2\lambda^2. \tag{50}$$

Hence $P(x) = -\lambda^2$ in (47) and two linearly independent solutions are

$$\Psi_1 = E(t)\mathrm{e}^{\lambda x} + F(t)\mathrm{e}^{-\lambda x} \qquad \Psi_2 = G(t)\mathrm{e}^{\lambda x} + H(t)\mathrm{e}^{-\lambda x}. \tag{51}$$

Therefore by lemma 1 and lemma 2 one obtains

$$\phi(t,x) = \frac{E(t)\mathrm{e}^{\lambda x} + F(t)\mathrm{e}^{-\lambda x}}{G(t)\mathrm{e}^{\lambda x} + H(t)\mathrm{e}^{-\lambda x}} \qquad E\ H - F\ G \neq 0. \tag{52}$$

To determine the t-dependence of $\phi(t,x)$ let us recall from (43) that

$$C = \beta\ S = -2\beta\lambda^2 = -\frac{\phi_t}{\phi_x}. \tag{53}$$

This leads to a system of nonlinear ordinary differential equations in the coefficients E, F, G and H:

$$\begin{cases} E\,G' - E'\,G = 0 \\ F\,H' - F'\,H = 0 \\ 4\beta\lambda^3 G\,H = G'\,H' - G\,H'. \end{cases} \tag{54}$$

A particular solution of (54) is

$$\frac{H(t)}{G(t)} = \mathrm{e}^{-4\beta\lambda^3 t} \qquad E(t) = B\,G(t) \qquad F(t) = A\,H(t)$$

where A, B $(A \neq B)$ are real arbitrary constants. This solution of (54) leads to

$$\phi(t,x) = A + B\mathrm{e}^{-\lambda\xi}\,\mathrm{sech}\,\lambda\xi \qquad \xi = x + 2\beta\lambda^2 t. \tag{55}$$

With this $\phi(t,x)$ one gets

$$u_0(t,x) = -\lambda A\sqrt{-\frac{\beta}{\alpha}}\mathrm{e}^{-\lambda\xi}\,\mathrm{sech}\,\lambda\xi\,(1+\tanh\lambda\xi) \tag{56}$$

$$u_1(t,x) = f_1(\xi) = \lambda\sqrt{-\frac{\beta}{\alpha}}\tanh\lambda\xi \tag{57}$$

and

$$u(t,x) = f_2(\xi) = \lambda\sqrt{-\frac{\beta}{\alpha}}\left(1 - \frac{2}{1+M\mathrm{e}^{2\lambda\xi}}\right) \tag{58}$$

where M is an arbitrary constant. For $M = 1$ one gets as a special case

$$u(t,x) = \lambda\sqrt{-\frac{\beta}{\alpha}}\coth\lambda\xi. \tag{59}$$

Hence $u(t,x)$ and $u_1(t,x)$ in the above are exact solutions for the usual MKdV equation

$$\Psi_t + 2\alpha(\Psi^3)_x + \beta\Psi_{xxx} = 0 \tag{60}$$

which are not new, and the corresponding exact solutions of the CMKdV equation (1) are

$$w(t,x) = u_1(t,x) + \mathrm{i}u_1(t,x)$$

and

$$w(t,x) = u(t,x) + \mathrm{i}u(t,x).$$

The functions $f_1(\xi)$, $f_2(\xi)$ in (57) and (58) are solutions of the similarity reduction

$$bf'' - cf + af^3 = D \tag{61}$$

of the MKdV equation (60) which is the similarity reduction (10) with real f's.

For $S = 2\lambda^2$. Equation (22) becomes

$$S = \{\phi; x\} = 2\lambda^2.$$

Hence $P(x) = \lambda^2$ in (47) and two linearly independent solutions are

$$V_1 = E(t)\mathrm{e}^{\lambda\mathrm{i}x} + F(t)\mathrm{e}^{-\lambda\mathrm{i}x} \qquad V_2 = G(t)\mathrm{e}^{\lambda\mathrm{i}x} + H(t)\mathrm{e}^{-\lambda\mathrm{i}x}.$$

By lemma 1 and lemma 2 one gets

$$\phi(t,x) = \frac{E(t)\mathrm{e}^{\lambda\mathrm{i}x} + F(t)\mathrm{e}^{-\lambda\mathrm{i}x}}{G(t)\mathrm{e}^{\lambda\mathrm{i}x} + H(t)\mathrm{e}^{-\lambda\mathrm{i}x}} \qquad EH - FG \neq 0. \tag{62}$$

To determine the t-dependence of $\phi(t,x)$ we use (53) and obtain

$$\phi(t,x) = A\tan\lambda\xi \qquad \xi = x + 2\beta\lambda^2 t \tag{63}$$

where A is an arbitrary constant. With this $\phi(t,x)$ one gets

$$u_0(t,x) = \lambda\sqrt{\frac{-\beta}{\alpha}}A\,\mathrm{sech}^2\,\lambda\xi \tag{64}$$

$$u_1(t,x) = f_3(\xi) = -\lambda\sqrt{\frac{-\beta}{\alpha}}\tan\lambda\xi \tag{65}$$

and

$$u(t,x) = f_4(\xi) = \lambda\sqrt{\frac{-\beta}{\alpha}}\cot\lambda\xi \tag{66}$$

where λ is an arbitrary constant.

The functions $u_1(t,x)$, $u(t,x)$ are well known solutions of the modified Korteweg–de Vries equation (60) and the corresponding solutions of the complex modified Korteweg–de Vries equation (1) are of the form

$$w(t,x) = u_1(t,x) + \mathrm{i}u_1(t,x)$$

and

$$w(t,x) = u(t,x) + \mathrm{i}u(t,x). \tag{67}$$

The functions $f_3(\xi)$, $f_4(\xi)$ in (65) and (66) are solutions of the similarity reduction in (61).

B. Solutions for $S = -4/x^2$. The function $S = -4/x^2$ is also a solution of the Korteweg–de Vries equation in (46).

With $S = -4/x^2$ equation (47) becomes

$$S = \{\phi; x\} = -\frac{4}{x^2}. \tag{68}$$

Hence $P(z) = -2/x^2$ in (47) and two linearly independent solutions are $1/x$ and x^2.

Therefore by lemma 1 and lemma 2 one obtains

$$\phi(t,x) = \frac{E(t)\,x^3 + F(t)}{G(t)\,x^3 + H(t)} \qquad E\,H - F\,G \neq 0. \tag{69}$$

To determine the t-dependence of $\phi(t,x)$ let us recall from (43) that

$$C = \beta\,S = -\frac{4\beta}{x^2} = -\frac{\phi_t}{\phi_x}. \tag{70}$$

This leads to a system of nonlinear ordinary differential equations.

$$\begin{cases} E\,F' - E'\,F = 0 \\ E\,H' - E'\,H + F\,G' - F'\,G = 0 \\ -12\beta(E\,H - F\,G) = F\,H' - F'\,H. \end{cases} \tag{71}$$

A particular solution of (71) can be found by inspection. Let E and G be constants; then one has

$$F(t) = 12E\beta\,t + M \qquad H(t) = 12G\beta\,t + N \tag{72}$$

where M and N are arbitrary constants. Hence one obtains

$$\phi(t,x) = 1 + \frac{A}{x^3 + 12\beta\, t + B}$$
$$u_0(t,x) = -\frac{3\varepsilon A\, x^2}{(x^3 + 12\beta\, t + B)^2} \qquad (73)$$
$$u_1(t,x) = \varepsilon \frac{2x^3 - 12\beta\, t - B}{(x^3 + 12\beta\, t + B)x}$$

where A and B are arbitrary constants.

The functions $u_1(t,x)$, $u(t,x)$ are well known solutions of the modified Korteweg–de Vries equation (60) and the corresponding solutions of the complex modified Korteweg–de Vries equation (1) is obtained by using (73) in (18)

$$w(t,x) = u(t,x) + \mathrm{i}v(t,x) \qquad (74)$$

with

$$u(t,x) = v(t,x) = f_5(\xi) = \sqrt{\frac{-\beta}{\alpha}}\left(\frac{3x^2}{x^3 + 12\beta\, t + K} - \frac{1}{x}\right) \qquad (75)$$

where K is an arbitrary constant.

For $K = 0$, the function $f_5(\xi)$ in (75) is a solution of another similarity reduction of the MKdV equation (60) which is the similarity reduction (9) with f is real and $D = (-2/3)\sqrt{-b/a}$.

Case 2. $S_{xx} \equiv 0$

In this case

$$S(t,x) = A(t)x + B(t). \qquad (76)$$

However, equations (34) and (35) imply that $A(t) \equiv 0$, and hence by the compatibility condition (25), $S = \pm 2\lambda^2$ is a real constant. By (36) $D = -\omega^2$ and by (32) $C = \beta(\pm 2\lambda^2 - 3\omega^2)$ are also real constants.

Following the same path of calculation as the one in the previous case one can find the travelling wave solutions of the complex modified Korteweg–de Vries equation.

For $S = 2\lambda^2$ one obtains

$$\phi(\xi) = a + b\tan\lambda\xi \qquad \xi = x - Ct$$
$$u_0(\xi) = \sqrt{-2\beta/\alpha}\; b\lambda\cos\omega\xi\;\sec^2\lambda\xi$$
$$v_0(\xi) = \sqrt{-2\beta/\alpha}\; b\lambda\sin\omega\xi\;\sec^2\lambda\xi \qquad (77)$$
$$u_1(\xi) = \sqrt{-2\beta/\alpha}\,(\omega\sin\omega\xi - \lambda\tan\lambda\xi\;\cos\omega\xi)$$
$$v_1(\xi) = -\sqrt{-2\beta/\alpha}\,(\omega\cos\omega\xi + \lambda\tan\lambda\xi\sin\omega\xi)$$

and hence

$$u_1(\xi) \qquad v_1(\xi)$$

as well as

$$u(\xi) = \frac{u_0(\xi)}{\phi(\xi)} + u_1(\xi) \qquad v(\xi) = \frac{v_0(\xi)}{\phi(\xi)} + v_1(\xi) \qquad (78)$$

give the travelling wave solutions of the system in (3). The corresponding solutions of the complex modified Korteweg–de Vries equation (1) are of the form

$$w(t,x) = f_6(\xi) = u_1(\xi) + \mathrm{i}v_1(\xi) \qquad (79)$$

and

$$w(t,x) = f_7(\xi) = u(\xi) + \mathrm{i}v(\xi). \tag{80}$$

Because of the dependence on the characteristic coordinate ξ, the $f_6(\xi)$ and $f_7(\xi)$ in the above are also solutions of the similarity reduction (10) of the complex modified Korteweg–de Vries equation.

Similarly for $S = -2\lambda^2$ one obtains

$$\begin{aligned}
\phi(\xi) &= a + b\tanh\lambda\xi \qquad \xi = x - Ct \\
u_0(\xi) &= \sqrt{-2\beta/\alpha}\; b\lambda\cos\omega\xi\;\operatorname{sech}^2\lambda\xi \\
v_0(\xi) &= \sqrt{-2\beta/\alpha}\; b\lambda\sin\omega\xi\;\operatorname{sech}^2\lambda\xi \\
u_1(\xi) &= \sqrt{-2\beta/\alpha}\,(\omega\sin\omega\xi - \lambda\tanh\lambda\xi\cos\omega\xi) \\
v_1(\xi) &= -\sqrt{-2\beta/\alpha}\,(\omega\cos\omega\xi + \lambda\tanh\lambda\xi\;\sin\omega\xi)
\end{aligned} \tag{81}$$

and hence (81) again gives another travelling wave solution of the system in (3). These produce another class of travelling wave solutions of the the complex modified Korteweg–de Vries equation (1) and hence another solution of the similarity reduction in (10).

6. Discussion

Painlevé analysis provides a new and powerful tool for constructing explicit solutions for non-integrable as well as integrable dynamical systems. But it only gives possible solutions, so one must check the results if they are actual solutions of the given nonlinear partial differential equation. On the other hand, the necessary calculations are in general too tedious to do by hand. In those cases one needs to use some computer algebra systems like Reduce or Mathematica.

References

[1] Erbay S and Şuhubî E S 1989 *Int. J. Eng. Sci.* **27** 895–914

[2] Erbay S and Şuhubî E S 1989 *Int. J. Eng. Sci.* **27** 915–9

[3] Weiss J, Tabor M and Carnevale 1983 *J. Math. Phys.* **24** 522–6

[4] Lie S 1891 *Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen* (Leipzig: Teuber) [reprinted 1967 (New York: Chelsea)]

[5] Bluman G W and Kumei S 1988 *Symmetries and Differential Equations* (Berlin: Springer)

[6] Osgood W F 1966 *Topics in the Theory of Functions of Several Complex Variables* (New York: Dover)

[7] Hille E 1966 *Ordinary Differential Equations in the Complex Plane* (New York: Wiley)

[8] Weiss J 1983 *J. Math. Phys.* **24** 1405–13

[9] Weiss J 1984 *J. Math. Phys.* **25** 13–24

[10] Weiss J 1985 *J. Math. Phys.* **26** 258–69

[11] Weiss J 1984 *J. Math. Phys.* **25** 2226–35

[12] Conte R and Musette M 1989 *J. Phys. A: Math. Gen.* **22** 169–77

[13] Steeb W H and Euler N 1988 *Nonlinear Evolution Equations and Painlevé Test* (Singapore: World Scientific)